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Problems of radially polarized piezoelastic bodies

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Abstract

In this paper, three displacement functions are introduced to simplify the basic equations of a radially polarized, spherically isotropic, piezoelectric medium. By expanding the displacement functions as well as the electric potential in terms of spherical harmonics, the basic equations are converted to an uncoupled Euler type, second-order ordinary differential equation and a coupled system of three such ones. Based on the well-known solution to the Euler equation, the general solution for the static problem is obtained. Some axisymmetric problems are then considered. It is noted that the present analysis is an extension of that of spherically isotropic pure elasticity (Chen, 1966). © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Piezoelectric ceramics and composites have been widely used in electromechanical devices and in smart material systems (Rao and Sunar, 1994) and, consequently, have attracted a lot of attentions from both scientists and technical engineers. Because of the difficulty related to the particular coupling effect between electric field and mechanical deformation, few problems were considered before 1990. Since 1990, however, a great number of works have been published, dealing with a variety of problems of cracks and inhomogeneities (Wang, 1992; Dunn and Wienecke, 1997; Huang, 1997), bending and vibration of plates and shells (Lee and Saravanos, 1997; Heyliger, 1997), general solutions and Green's functions (Ding et al., 1996; Dunn and Wienecke, 1996; Ding et al., 1997) among others. Because the most technologically important PZMs are poled ceramics which exhibit transverse isotropy with the unique axis aligned along the poling direction, the above mentioned works have taken the effect of transverse isotropy into consideration. It is noted that when the piezoelectric materials are poled spherically in the radial direction, they will exhibit spherical isotropy, of which the linear constitutive relations can be expressed as follows [assuming the center of anisotropy coincides with the origin of the spherical coordinates (r, θ, ϕ)]:

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$$\sigma_{\theta\theta} = c_{11}s_{\theta\theta}+c_{12}s_{\phi\phi}+c_{13}s_{rr}+e_{31}\frac{\partial\Phi}{\partial r}, \quad \sigma_{r\theta} = 2c_{44}s_{r\theta}+\frac{e_{15}}{r}\frac{\partial\Phi}{\partial\theta},$$

$$\sigma_{\phi\phi} = c_{12}s_{\theta\theta}+c_{11}s_{\phi\phi}+c_{13}s_{rr}+e_{31}\frac{\partial\Phi}{\partial r}, \quad \sigma_{r\theta} = 2c_{44}s_{r\phi}+\frac{e_{15}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi},$$

$$\sigma_{rr} = c_{13}s_{\theta\theta}+c_{13}s_{\phi\phi}+c_{33}s_{rr}+e_{33}\frac{\partial\Phi}{\partial r}, \quad \sigma_{\theta\phi} = (c_{11}-c_{12})s_{\theta\phi},$$

$$D_\theta = 2e_{15}s_{r\theta}-\frac{\varepsilon_{11}}{r}\frac{\partial\Phi}{\partial\theta}, \quad D_\theta = 2e_{15}s_{r\phi}-\frac{\varepsilon_{11}}{r\sin\theta}\frac{\partial\Phi}{\partial\phi},$$

$$D_r = e_{31}s_{\theta\theta}+e_{31}s_{\phi\phi}+e_{33}s_{rr}-\varepsilon_{33}\frac{\partial\Phi}{\partial r}, \tag{1}$$

where, σ_{ij} and s_{ij} are the stress and strain tensors, respectively; Φ and D_i are the electric potential and electric displacement vector, respectively; c_{ij} are the elastic stiffness constants (measured in a constant electric field), ε_{ij} are the dielectric constants (measured at constant strain), and e_{ij} the piezoelectric constants. The strain and mechanical displacement relations and the equilibrium equations are the same as those of pure elasticity so that they are not given here for the sake of simplicity. The reader is referred to the excellent monograph by Lekhnitskii (1981). The charge equation of electrostatics is (Tiersten, 1969):

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} = \rho_{\mathrm{f}}, \tag{2}$$

where ρ_{f} is the free charge density. Despite the fact that the spherical configuration is very common in electromechanical devices (Kirichok, 1980), few theoretical works on the subject can be found in literature.

Even for pure elasticity, the basic equations of a spherically isotropic body seem too complicated to be solved directly. To simplify the basic equations of equilibrium, Hu (1954) introduced two displacement potentials to represent displacement components; he showed that the general solutions may be found through the use of spherical harmonics. On the basis of Hu's separation method, Chen (1966) considered some axisymmetric problems such as a concentrated force in an infinite medium, stress concentration due to a spherical cavity, and a steadily rotating shell. Recently, Chen (1995) simplified the equations of motion of a spherically isotropic elastic medium with radial inhomogeneity by adopting three displacement functions and considered some coupled vibration problems of spherical shells (Ding and Chen, 1996a, b).

In this paper, three displacement functions (Chen, 1995; Ding and Chen, 1996a, b) are employed to represent three displacement components. It is found that the basic equations of equilibrium and the charge equation of a spherically isotropic body are then reduced to an uncoupled partial differential equation and a coupled system of three partial differential equations. Some considerations on the solution are then presented. As expected, for the usual transversely isotropic piezoelectricity, results degenerate to those available in the literature. By expanding three displacement functions as well as the electric potential in terms of spherical harmonics, the controlling equations are further simplified to an uncoupled second-order ordinary differential equation (SOODE) and a coupled system of three SOODEs. All these SOODEs are of the Euler type and their solutions can be readily obtained. A general solution for the static problem is then derived, and some axisymmetric problems are considered. Numerical results are given for concentrations of stress and electric displacement in the neighborhood of a spherical cavity to show the effects of material constants. The distributions of nondimensional hoop stress $\sigma_{\theta\theta}$ and electric displacement D_θ along the radius in the vicinity of the cavity are also presented in figure form.

2. Separation technique

2.1. The decomposition of mechanical displacements

Three displacement functions w, G and ψ are now introduced so that the three components of mechanical displacement u_r, u_θ and u_ϕ are expressed as follows (Chen, 1995; Ding and Chen, 1996a, b)

$$u_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}-\frac{\partial G}{\partial\theta},\quad u_\phi=\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial G}{\partial\phi},\quad u_r=w. \tag{3}$$

It is noted here that the present use of three displacement functions is much simpler than that of two potential functions employed by other authors mentioned earlier. It also seems natural that three displacement components are represented by three displacement functions so that it is easier to understand. Suppose the body force components F_i $(i=r,\theta,\phi)$ can also be decomposed in the same way, i.e.

$$rF_\theta = -\frac{1}{\sin\theta}\frac{\partial V}{\partial\phi}-\frac{\partial U}{\partial\theta},\quad rF_\phi=\frac{\partial V}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}. \tag{4}$$

The most common case is that the body force vector is potential, then

$$V=0,\quad F_r=-\frac{\partial U}{\partial r}. \tag{5}$$

By employing eqns (3)–(4), through some lengthy manipulations, we can transfer the basic equations of a spherically isotropic piezoelectric body to the following equations:

$$\frac{\partial}{\partial\theta}[A-rU]-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}[B+rV]=0,\quad \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}[A-rU]+\frac{\partial}{\partial\theta}[B+rV]=0, \tag{6}$$

$$L_3w-L_4\nabla_1^2G+L_5\Phi+r^2F_r=0, \tag{7}$$

$$L_7w-L_8\nabla_1^2G-L_9\Phi=r^2\rho_{\mathrm{f}}, \tag{8}$$

where

$$A=L_1w-L_2G+L_6\Phi,\quad B=[c_{44}\nabla_3^2-2c_{44}+c_{11}-c_{12}+\tfrac{1}{2}(c_{11}-c_{12})\nabla_1^2]\psi,$$

$$L_1=(c_{13}+c_{44})\nabla_2+c_{11}+c_{12}+2c_{44},\quad L_2=c_{44}\nabla_3^2-2c_{44}+c_{11}-c_{12}+c_{11}\nabla_1^2,$$

$$L_3=c_{33}\nabla_3^2-2(c_{11}+c_{12}-c_{13})+c_{44}\nabla_1^2,\quad L_4=(c_{13}+c_{44})\nabla_2-c_{44}-c_{11}-c_{12}+c_{13},$$

$$L_5 = e_{33}\nabla_3^2 - 2e_{31}\nabla_2 + e_{15}\nabla_1^2, \quad L_6 = (e_{15}+e_{31})\nabla_2 + 2e_{15},$$
$$L_7 = e_{33}\nabla_3^2 + 2e_{31}\nabla_2 + 2e_{31} + e_{15}\nabla_1^2, \quad L_8 = (e_{31}+e_{15})\nabla_2 + e_{31} - e_{15},$$
$$L_9 = \varepsilon_{33}\nabla_3^2 + \varepsilon_{11}\nabla_1^2, \quad \nabla_2 = r\frac{\partial}{\partial r}, \quad \nabla_2^2 = r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}, \quad \nabla_3^2 = \nabla_2^2 + \nabla_2,$$
$$\nabla_1^2 = \frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \tag{9}$$

From eqns (6), one can obtain:

$$A - rU = \frac{\partial H}{\partial \phi}, \quad B + rV = \sin\theta\frac{\partial H}{\partial \theta}. \tag{10}$$

Substituting eqns (10) into eqns (6) yields $\nabla_1^2 H = 0$. Generally, we can assume $H \equiv 0$. For details, see for examples, Hu (1954) and Chen (1995) for homogeneous and non-homogeneous spherically isotropic elasticity, respectively. Similar demonstration can be given for the piezoelectric case, however, it is omitted here because no unexpected difficulty is involved. Under this consideration, eqns (10) read

$$A - rU = 0, \quad B + rV = 0. \tag{11,12}$$

It is seen that function ψ is uncoupled from the other two displacement functions w and G, and the electric potential Φ. In particular, eqn (12) is a second order, uncoupled partial differential equation in ψ; eqns (7), (8) and (11) form a coupled partial differential equation system in w, G and Φ. The separability of the controlling equations of spherically isotropic, piezoelectric elasticity will facilitate the solution to relevant problems, as shown later.

2.2. General considerations on the solution

The solution to eqn (12) can be written as:

$$\psi = \psi_0 + \psi_1, \tag{13}$$

where ψ_0 is the general solution of the following homogeneous equation:

$$[c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11}-c_{12})\nabla_1^2]\psi_0 = 0 \tag{14}$$

and ψ_1 is the particular solution of the associated non-homogeneous equation:

$$[c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11}-c_{12})\nabla_1^2]\psi_1 = -rV. \tag{15}$$

The solution to the coupled system can be written as

$$w = \sum_{i=0}^{3} w_i, \quad G = \sum_{i=0}^{3} G_i, \quad \Phi = \sum_{i=0}^{3} \Phi_i, \tag{16}$$

where (w_0, G_0, Φ_0) and (w_i, G_i, Φ_i), $(i = 1, 2, 3)$ are the general and particular solutions to the corresponding homogeneous and inhomogeneous equations, respectively. They can be expressed as:

$$w_0 = A_{i1}F, \quad G_0 = A_{i2}F, \quad \Phi_0 = A_{i3}F, \quad (i = 1, 2, 3), \tag{17}$$

$$w_j = A_{j1}F_j^*, \quad G_j = A_{j2}F_j^*, \quad \Phi_j = A_{j3}F_j^*, \quad (j = 1, 2, 3, \text{no summation}), \tag{18}$$

where A_{ij} $(i, j = 1, 2, 3)$ are cofactors of the determinant $|D|$. Here D is the following operator matrix

$$D = \begin{bmatrix} L_1 & -L_2 & L_6 \\ L_3 & -L_4\nabla_1^2 & L_5 \\ L_7 & -L_8\nabla_1^2 & -L_9 \end{bmatrix} \tag{19}$$

F and $F_j^*(j = 1, 2, 3)$ satisfy the following homogeneous and non-homogeneous equations, respectively,

$$|D|F = 0, \quad |D|F_1^* = rU, \quad |D|F_2^* = -r^2F_r, \quad |D|F_3^* = r^2\rho_{\mathrm{f}}. \tag{20}$$

2.3. Transverse isotropy

The transverse isotropy case usually described in the cylindrical coordinates (r_1, ϕ, z) or Cartesian coordinates (x, y, z) can be seen as a limiting case of the spherical isotropy by the following limiting procedure:

$$r\sin\theta \to r_1, \quad \cos\theta \to 1, \quad \frac{\partial}{r\,\partial\theta} \to \frac{\partial}{\partial r_1}, \quad \frac{\partial}{\partial r} \to \frac{\partial}{\partial z}, \quad u_\theta \to u_{r_1}, \quad u_r \to u_z,$$

$$\frac{1}{r^2}\nabla_1^2 \to \frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2}{\partial\phi^2} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Lambda. \tag{21}$$

Now assuming $\psi = -\psi^*/r$ and $G = G^*/r$ in eqn (3), and making use of eqn (21), one obtains

$$u_{r_1} = \frac{\partial\psi^*}{r_1\,\partial\phi} - \frac{\partial G^*}{\partial r_1}, \quad u_\phi = -\frac{\partial\psi^*}{\partial r_1} - \frac{\partial G^*}{r_1\,\partial\phi}, \quad u_z = w, \tag{22}$$

which has been shown by Ding et al. (1996). We can also write down eqn (4) in this case as

$$F_{r_1} = \frac{\partial V^*}{r_1\,\partial\phi} - \frac{\partial U}{\partial r_1}, \quad F_\phi = -\frac{\partial V^*}{\partial r_1} - \frac{\partial U}{r_1\,\partial\phi}, \tag{23}$$

where $V = -V^*$ was introduced. Thus eqn (12) will take the following form for transverse isotropy:

$$\left[c_{44}\frac{\partial^2}{\partial z^2} + \frac{1}{2}(c_{11} - c_{12})\Lambda\right]\psi^* + V^* = 0, \tag{24}$$

which is the same as eqn (19) in Ding et al. (1996), where the effect of body force was absent. By virtue of eqns (21) and (22), the coupled system of eqns (7), (8) and (11) becomes:

$$\begin{bmatrix} (c_{13}+c_{44})\dfrac{\partial}{\partial z} & -\left(c_{11}\Lambda+c_{44}\dfrac{\partial^2}{\partial z^2}\right) & (e_{15}+e_{31})\dfrac{\partial}{\partial z} \\ c_{44}\Lambda+c_{33}\dfrac{\partial^2}{\partial z^2} & -(c_{13}+c_{44})\Lambda\dfrac{\partial}{\partial z} & e_{15}\Lambda+e_{33}\dfrac{\partial^2}{\partial z^2} \\ e_{15}\Lambda+e_{33}\dfrac{\partial^2}{\partial z^2} & -(e_{15}+e_{31})\Lambda\dfrac{\partial}{\partial z} & -\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\dfrac{\partial^2}{\partial z^2}\right) \end{bmatrix} \begin{Bmatrix} w \\ G^* \\ \Phi \end{Bmatrix} = \begin{Bmatrix} U \\ -F_z \\ \rho_{\mathrm{f}} \end{Bmatrix}, \tag{25}$$

where the property $F_r \to F_z$ was employed. Omitting the right hand side, eqn (25) is found to be the same as eqn (20) in Ding et al. (1996). Solution to eqn (25) can be constructed by using operator theory and the reader is referred to Ding et al. (1996).

3. General nonaxisymmetric solution

Noticing that all resulting equations include the partial operator ∇_1^2 which is defined in eqn (9), we assume

$$\psi = \sum_{n=1}^{\infty} \psi_n(r) S_n^m(\theta,\phi), \quad w = \sum_{n=0}^{\infty} w_n(r) S_n^m(\theta,\phi),$$

$$G = \sum_{n=1}^{\infty} G_n(r) S_n^m(\theta,\phi), \quad \Phi = \frac{e_{33}}{\varepsilon_{33}} \sum_{n=0}^{\infty} \Phi_n(r) S_n^m(\theta,\phi), \tag{26}$$

where $S_n^m(\theta,\phi) = P_n^m(\cos\theta)\exp(im\phi)$ are spherical harmonics and $P_n^m(\cos\theta)$ are the associated Legendre functions; n and m are integers. Substitution of eqn (26) into eqns (7), (8), (11) and (12) yields

$$r^2\psi_n''+2r\psi_n'-[2+(n-1)(n+2)(f_1-f_2)/2]\psi_n = 0, \tag{27}$$

$$r^2 w_n''+2rw_n'+p_1 w_n-p_2 rG_n'-p_3 G_n+q_1 r^2\Phi_n''+q_2 r\Phi_n'+q_3\Phi_n = 0, \tag{28}$$

$$r^2 G_n''+2rG_n'+p_4 G_n-p_5 rw_n'-p_6 w_n+q_4 r\Phi_n'+q_5\Phi_n = 0, \tag{29}$$

$$r^2\Phi_n''+2r\Phi_n'+q_6\Phi_n-r^2 w_n''-p_7 rw_n'-p_8 w_n-p_9 rG_n'-p_{10}G_n = 0, \tag{30}$$

where a prime stands for differentiation with respect to r, and

$$p_1 = [2(f_3-f_1-f_2)-n(n+1)]/f_4, \quad p_2 = -n(n+1)(f_3+1)/f_4,$$

$$p_3 = n(n+1)(f_1+f_2+1-f_3)/f_4, \quad p_4 = f_1-f_2-n(n+1)f_1-2,$$

$$p_5 = f_3+1, \quad p_6 = f_1+f_2+2, \quad p_7 = 2(f_6+1), \quad p_8 = 2f_6-n(n+1)f_5,$$

$$p_9 = n(n+1)(f_5+f_6), \quad p_{10} = n(n+1)(f_6-f_5), \quad q_1 = f_8/f_4,$$

$$q_2 = 2f_8(1-f_6)/f_4, \quad q_3 = -n(n+1)f_5 f_8/f_4, \quad q_4 = -(f_5+f_6)f_8,$$

$$q_5 = -2f_5 f_8, \quad q_6 = -n(n+1)f_7,$$

$$f_1 = c_{11}/c_{44},\quad f_2 = c_{12}/c_{44},\quad f_3 = c_{13}/c_{44},\quad f_4 = c_{33}/c_{44},$$
$$f_5 = e_{15}/e_{33},\quad f_6 = e_{31}/e_{33},\quad f_7 = \varepsilon_{11}/\varepsilon_{33},\quad f_8 = e_{33}^2/(\varepsilon_{33}c_{44}). \tag{31}$$

Notice here that the body force and free charge density have been dropped during the derivation. It is obvious that solutions to the so-called Euler eqns (27)–(30) can be obtained by assuming

$$G_n = A_n r^{v_n-1/2},\quad w_n = B_n r^{v_n-1/2},\quad \Phi_n = C_n r^{v_n-1/2},\quad \psi_n = D_n r^{\lambda_n-1/2}, \tag{32}$$

where A_n, B_n, C_n and D_n are undetermined constants. Substituting eqn (32) into eqns (27)–(30) yields

$$\{\lambda_n^2-[9/4+(n-1)(n+2)(f_1-f_2)/2]\}D_n = 0, \tag{33}$$

$$\mathbf{H}\begin{Bmatrix} A_n \\ B_n \\ C_n \end{Bmatrix} = 0, \tag{34}$$

where,

$$\mathbf{H} = \begin{bmatrix} v_n^2-\frac{1}{4}+p_4 & -p_5(v_n-\frac{1}{2})-p_6 & q_4(v_n-\frac{1}{2})+q_5 \\ -p_2(v_n-\frac{1}{2})-p_3 & v_n^2-\frac{1}{4}+p_1 & q_1(v_n^2-\frac{1}{4})+(q_2-2q_1)(v_n-\frac{1}{2})+q_3 \\ -p_9(v_n-\frac{1}{2})-p_{10} & -v_n^2+\frac{1}{4}-(p_7-2)(v_n-\frac{1}{2})-p_8 & v_n^2-\frac{1}{4}+q_6 \end{bmatrix}. \tag{35}$$

Apparently, from eqn (33), one can get

$$\lambda_{n1,2} = \pm[9/4+(n-1)(n+2)(f_1-f_2)/2]^{1/2}. \tag{36}$$

It is seen that eqn (36) is identical to that for pure elasticity (Chen, 1966). From eqns (34), we can obtain the equation governing v_n as

$$|\mathbf{H}| = 0. \tag{37}$$

It can be verified that eqn (37) is a cubic algebraic one in v_n^2. For stable materials, the eigenvalue v_n of eqn (37) cannot be pure imaginary. Thus we assume that $v_{ni} = -v_{n(i+3)}$ and $\mathrm{Re}\,[v_{ni}] < 0$ $(i = 1, 2, 3)$ and that six eigenvalues of eqn (37) are distinct (if there are repeated roots, solutions (32) shall take logarithm form). If rank($\mathbf{H}$) = 2, one obtains the following relations from eqns (34):

$$B_{ni} = K_{ni}^1 A_{ni},\quad C_{ni} = K_{ni}^2 A_{ni}, \tag{38}$$

for each eigenvalue v_{ni}, where K_{ni}^1 and K_{ni}^2 can be solved from the two independent equations in eqns (34). Making use of eqns (3), (26), (32) and (38), we obtain a general solution to the basic equations as follows:

$$u_r = \sum_{n=0}^{\infty}\sum_{i=1}^{6} K_n^1 A_{ni} r^{v_{ni}-1/2} S_n^m(\theta,\phi),$$

$$u_\theta = -\sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni} r^{v_{ni}-1/2} \frac{\partial}{\partial\theta} S_n^m(\theta,\phi) - \sum_{n=1}^{\infty}\sum_{i=1}^{2} D_{ni} r^{\lambda_{ni}-1/2} \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} S_n^m(\theta,\phi),$$

$$u_\phi = -\sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni} r^{v_{ni}-1/2} \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} S_n^m(\theta,\phi) + \sum_{n=1}^{\infty}\sum_{i=1}^{2} D_{ni} r^{\lambda_{ni}-1/2} \frac{\partial}{\partial\theta} S_n^m(\theta,\phi),$$

$$\Phi = \frac{e_{33}}{\varepsilon_{33}} \sum_{n=0}^{\infty}\sum_{i=1}^{6} K_{ni}^2 A_{ni} r^{v_{ni}-1/2} S_n^m(\theta,\phi). \quad (39)$$

It is noted here that eqns (27)–(30) are only valid for $n \geqslant 1$, while for $n = 0$ we have, instead the following two equations:

$$r^2 w_0'' + 2r w_0' + p_1 w_0 + q_1 r^2 \Phi_0'' + q_2 r \Phi_0' = 0, \quad (40)$$

$$r^2 \Phi_0'' + 2r\Phi_0' - r^2 w_0'' - p_7 r w_0' - p_8 w_0 = 0. \quad (41)$$

As a result, we obtain a fourth-order eigenequation instead of eqn (37), the sixth-order one. To write in a united form as shown in eqn (39), we shall employ the following formulae for $n = 0$ there:

$$A_{0i} = C_{0i}, \quad K_{0i}^2 = 1, \quad K_{0i}^1 = -\frac{q_1(v_{0i}^2 - 1/4) + (q_2 - 2q_1)(v_{0i} - 1/2)}{v_{0i}^2 - 1/4 + p_1}, \quad (i = 1,2,4,5) \quad (42)$$

and $A_{03} = A_{06} = 0$. It can be shown that the solutions corresponding to $v_{04} = 1/2$, $v_{14} = 1/2$ and $\lambda_{12} = 3/2$ all give zero stress fields as well as zero electric displacements. For the sake of convenience, we assumed that A_{04}, A_{14}, and B_{12} are all equal to zero.

If the piezoelectric effect is neglected, expressions (39) degenerate identically to those of elasticity (Chen, 1966).

4. Axisymmetric problems

4.1. The general solution

As considered by Chen (1966), we shall pay attention to the boundary-value problem when the piezoelectric medium is bounded by two concentric spherical surfaces defined by r_1 and r_2, $0 \leqslant r_1 \leqslant r_2$. For the axisymmetric problem, eqns (39) read

$$u_r = \sum_{n=0}^{\infty}\sum_{i=1}^{6} K_{ni}^1 A_{ni} r^{v_{ni}-1/2} P_n(\cos\theta), \quad u_\theta = -\sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni} r^{v_{ni}-1/2} \frac{\partial}{\partial\theta} P_n(\cos\theta),$$

$$u_\phi = \sum_{n=1}^{\infty}\sum_{i=1}^{2} D_{ni} r^{\lambda_{ni}-1/2} \frac{\partial}{\partial\theta} P_n(\cos\theta), \quad \Phi = \frac{e_{33}}{\varepsilon_{33}} \sum_{n=0}^{\infty}\sum_{i=1}^{6} K_{ni}^2 A_{ni} r^{v_{ni}-1/2} P_n(\cos\theta), \quad (43)$$

where $P_n(\cos\theta)$ is the Legendre polynomial. The stress and electric displacement components are:

$$
\begin{aligned}
\sigma_{rr} &= \sum_{n=0}^{\infty}\sum_{i=1}^{6} A_{ni}\left\{c_{33}K_{ni}^{1}\left(v_{ni}-\frac{1}{2}\right)+c_{13}[2K_{ni}^{1}+n(n+1)]+\frac{e_{33}^{2}}{\varepsilon_{33}}K_{ni}^{2}\left(v_{ni}-\frac{1}{2}\right)\right\}r^{v_{ni}-3/2}P_n(\cos\theta),\\
\sigma_{r\theta} &= \sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni}\left\{c_{44}\left[K_{ni}^{1}-\left(v_{ni}-\frac{3}{2}\right)\right]+\frac{e_{15}e_{33}}{\varepsilon_{33}}K_{ni}^{2}\right\}r^{v_{ni}-3/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
\sigma_{r\phi} &= \sum_{n=1}^{\infty}\sum_{i=1}^{2} c_{44}D_{ni}\left(\lambda_{ni}-\frac{3}{2}\right)r^{\lambda_{ni}-3/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
\sigma_{\theta\theta}+\sigma_{\phi\phi} &= \sum_{n=0}^{\infty}\sum_{i=1}^{6} A_{ni}\left\{(c_{11}+c_{12})[2K_{ni}^{1}+n(n+1)]+2\left(v_{ni}-\frac{1}{2}\right)\right.\\
&\qquad\times\left.\left(c_{13}K_{ni}^{1}+\frac{e_{31}e_{33}}{\varepsilon_{33}}K_{ni}^{2}\right)\right\}r^{v_{ni}-3/2}P_n(\cos\theta),\\
\sigma_{\theta\theta}-\sigma_{\phi\phi} &= \sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni}(c_{11}-c_{12})\left[n(n+1)P_n(\cos\theta)+2\cot\theta\frac{\partial}{\partial\theta}P_n(\cos\theta)\right]r^{v_{ni}-3/2},\\
\sigma_{\theta\phi} &= -\sum_{n=1}^{\infty}\sum_{i=1}^{2}\frac{1}{2}(c_{11}-c_{12})D_{ni}r^{\lambda_{ni}-3/2}\left[n(n+1)P_n(\cos\theta)+2\cot\theta\frac{\partial}{\partial\theta}P_n(\cos\theta)\right],\\
D_{\theta} &= \sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni}\left\{e_{15}\left[K_{ni}^{1}-\left(v_{ni}-\frac{3}{2}\right)\right]-\frac{\varepsilon_{11}e_{33}}{\varepsilon_{33}}K_{ni}^{2}\right\}r^{v_{ni}-3/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
D_{\phi} &= \sum_{n=1}^{\infty}\sum_{i=1}^{2} e_{15}D_{ni}\left(\lambda_{ni}-\frac{3}{2}\right)r^{\lambda_{ni}-3/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
D_{r} &= \sum_{n=0}^{\infty}\sum_{i=1}^{6} A_{ni}\left\{e_{33}K_{ni}^{1}\left(v_{ni}-\frac{1}{2}\right)+e_{31}[2K_{ni}^{1}+n(n+1)]-e_{33}K_{ni}^{2}\left(v_{ni}-\frac{1}{2}\right)\right\}r^{v_{ni}-3/2}P_n(\cos\theta).
\end{aligned}
\tag{44}
$$

We can establish without difficulty the corresponding relations for the boundary conditions as for the case for elasticity (Chen, 1966) and derive the linear equations to determine the arbitrary constants A_{ni} and D_{ni} for the general axisymmetric problem. However, these are omitted and some axisymmetric problems will be considered in the following:

4.2. Concentrated force P acting at the origin

From the following mechanical equilibrium conditions over any surface enclosing the origin

$$2\pi\int_0^{\pi}(\sigma_r\cos\theta-\tau_{r\theta}\sin\theta)r^2\sin\theta\,\mathrm{d}\,\theta+P=0, \tag{45}$$

it can be seen that the stress must include factor r^{-2}. After considerable computation, it is shown that eqn (37) has a root $-1/2$ for $n = 1$. Without loss of the generality, we take $v_{11} = -1/2$ and eqns (34) give

$$K_{11}^1 = \frac{f_7(f_1+f_2+2)-2(f_6-f_5)f_5f_8}{f_7(f_3-f_1-f_2-1)+(f_6-f_5)f_5f_8}, \quad K_{11}^2 = \frac{f_5(2f_3-f_1-f_2)}{f_7(f_3-f_1-f_2-1)+(f_6-f_5)f_5f_8}. \tag{46}$$

Thus the non-zero piezoelectric fields due to a concentrated force P at the origin can be derived:

$$\sigma_{rr} = -\frac{3P}{4\pi}\frac{K_{11}^1(f_4-2f_3)-2f_3+f_8K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\cos\theta}{r^2},$$

$$\sigma_{r\theta} = -\frac{3P}{4\pi}\frac{K_{11}^1+2+f_5f_8K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\sin\theta}{r^2},$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{3P}{4\pi}\frac{(f_1+f_2)(K_{11}^1+1)-f_3K_{11}^1-f_6f_8K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\cos\theta}{r^2},$$

$$D_r = -\frac{3Pe_{33}}{4\pi c_{44}}\frac{K_{11}^1(1-2f_6)-2f_6-K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\cos\theta}{r^2},$$

$$D_\theta = \frac{3Pe_{33}}{4\pi c_{44}}\frac{-f_5(K_{11}^1+2)+f_7K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\sin\theta}{r^2},$$

$$u_r = \frac{3P}{4\pi c_{44}}\frac{K_{11}^1}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\cos\theta}{r},$$

$$u_\theta = \frac{3P}{4\pi c_{44}}\frac{1}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\sin\theta}{r},$$

$$\Phi = \frac{3Pe_{33}}{4\pi c_{44}\varepsilon_{33}}\frac{K_{11}^2}{K_{11}^1(f_4-2f_3-2)+K_{11}^2(1-2f_5)f_8-2f_3-4}\frac{\cos\theta}{r}. \tag{47}$$

At this stage, one can easily derive the results corresponding to elasticity from the foregoing equations. It is found there are some printing errors in eqns (28) and (29) in Chen (1966).

4.3. Point charge Q acting at the origin

The electric equilibrium condition over any surface enclosing the origin demands

$$2\pi\int_0^\pi D_r r^2 \sin\theta \, d\theta = Q. \tag{48}$$

Allowing for it, the non-zero expressions of the piezoelectric field due to a point charge Q acting at the origin are derived:

$$\sigma_{rr} = \frac{Qc_{44}}{4\pi e_{33}} \frac{K_{01}^1(2f_3 - f_4) - f_8}{K_{01}^1(2f_6 - 1) + 1} \frac{1}{r^2}, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{Qc_{44}}{4\pi e_{33}} \frac{K_{01}^1(f_1 + f_2 - f_3) - f_6 f_8}{K_{01}^1(2f_6 - 1) + 1} \frac{1}{r^2},$$

$$D_r = \frac{Q}{4\pi} \frac{1}{r^2}, \quad u_r = \frac{Q}{4\pi e_{33}} \frac{K_{01}^1}{K_{01}^1(2f_6 - 1) + 1} \frac{1}{r}, \quad \Phi = \frac{Q}{4\pi \varepsilon_{33}} \frac{1}{K_{01}^1(2f_6 - 1) + 1} \frac{1}{r}, \tag{49}$$

where $K_{01}^1 = f_6 f_8/(f_1 + f_2 - f_3)$.

4.4. Stress concentration in the neighborhood of a spherical cavity

Here we intend to give the piezoelectric field of an infinite piezoelastic medium with a spherical cavity ($r = r_1$) due to a uniform tension σ_{zz}^0 applied at infinity. We shall find a supplementary solution which vanishes at infinity and which satisfies the following boundary conditions at $r = r_1$:

$$\sigma_{rr} = -\sigma_{zz}^0 \cos^2\theta = -\left[\frac{1}{3} + \frac{2}{3} P_2(\cos\theta)\right] \sigma_{zz}^0,$$

$$\sigma_{r\theta} = \sigma_{zz}^0 \sin\theta \cos\theta = -\frac{\sigma_{zz}^0}{3} \frac{\partial}{\partial\theta} P_2(\cos\theta),$$

$$D_r = 0. \tag{50}$$

By comparing eqns (50) and (44), we get the equations to determine the nonzero constants A_{01}, A_{02}, A_{21}, A_{22} and A_{23} as follows:

$$\sum_{i=1}^{2} A_{0i} r_1^{v_{0i} - 3/2} \left[c_{33} K_{0i}^1 \left(v_{0i} - \frac{1}{2}\right) + 2c_{13} K_{0i}^1 + \frac{e_{33}^2}{\varepsilon_{33}} K_{0i}^2 \left(v_{0i} - \frac{1}{2}\right)\right] = -\frac{1}{3}\sigma_{zz}^0,$$

$$\sum_{i=1}^{2} A_{0i} r_1^{v_{0i} - 3/2} \left[e_{33} K_{0i}^1 \left(v_{0i} - \frac{1}{2}\right) + 2e_{31} K_{0i}^1 - e_{33} K_{0i}^2 \left(v_{0i} - \frac{1}{2}\right)\right] = 0.$$

$$\sum_{i=1}^{3} A_{2i} r_1^{v_{2i} - 3/2} \left[c_{33} K_{2i}^1 \left(v_{2i} - \frac{1}{2}\right) + 2c_{13}(K_{2i}^1 + 3) + \frac{e_{33}^2}{\varepsilon_{33}} K_{2i}^2 \left(v_{2i} - \frac{1}{2}\right)\right] = -\frac{2}{3}\sigma_{zz}^0, \tag{51}$$

$$\sum_{i=1}^{3} A_{2i} r_1^{v_{2i} - 3/2} \left\{c_{44}\left[K_{2i}^1 - \left(v_{2i} - \frac{3}{2}\right)\right] + \frac{e_{15} e_{33}}{\varepsilon_{33}} K_{2i}^2\right\} = -\frac{1}{3}\sigma_{zz}^0,$$

$$\sum_{i=1}^{3} A_{2i} r_1^{v_{2i} - 3/2} \left[e_{33} K_{2i}^1 \left(v_{2i} - \frac{1}{2}\right) + 2e_{31}(K_{2i}^1 + 3) - e_{33} K_{2i}^2 \left(v_{2i} - \frac{1}{2}\right)\right] = 0. \tag{52}$$

The total piezoelectric field is then obtained by combining the above supplementary solution with that caused by the uniform tension in a perfect piezoelectric body. As done in Chen (1966), it is interesting to give the concentration factor defined by $\sigma_{\theta\theta}(r_1, 90^\circ, \phi)/\sigma_{zz}^0$ as follows:

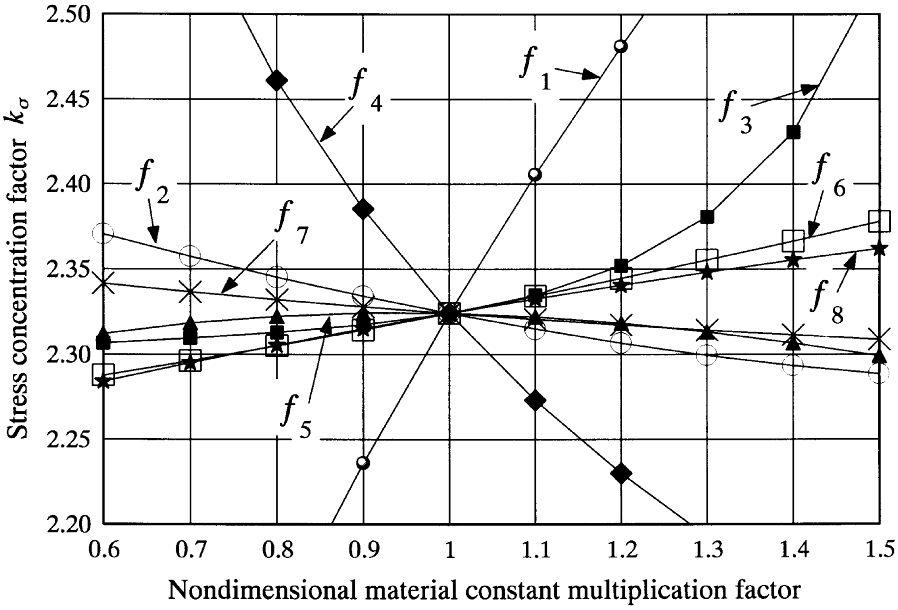

Fig. 1. Stress concentration factor when one of the nondimensional material constants (PZT-4) is multiplied by a factor from 0.6–1.5.

$$
\begin{aligned}
k_\sigma &= \sigma_{\theta\theta}(r_1, 90^\circ, \phi)/\sigma_{zz}^0 \\
&= 1 + \sum_{i=1}^{2} (c_{44} A_{0i} r_1^{\nu_{0i}-3/2}/\sigma_{zz}^0) \left\{ K_{0i}^1 \left[f_1 + f_2 + f_3 \left(\nu_{0i} - \frac{1}{2} \right) \right] + f_6 f_8 K_{0i}^2 \left(\nu_{0i} - \frac{1}{2} \right) \right\} \\
&\quad - \frac{1}{2} \sum_{i=1}^{3} (c_{44} A_{2i} r_1^{\nu_{2i}-3/2}/\sigma_{zz}^0) \left\{ K_{2i}^1 \left[f_1 + f_2 + f_3 \left(\nu_{2i} - \frac{1}{2} \right) \right] + 6 f_1 + f_6 f_8 K_{2i}^2 \left(\nu_{2i} - \frac{1}{2} \right) \right\}. \qquad (53)
\end{aligned}
$$

It can be seen from eqns (51)–(53) that the stress concentration factor k_σ is independent of the radius r_1 as for elastic materials.

The effects of nondimensional material constants on the stress concentration factor k_σ are shown in Fig. 1. Each curve gives the stress concentration factor when one of the material constants is multiplied by a factor from 0.6–1.5 for PZT-4, the constants of which (and other piezoelectric materials considered hereafter) can be found in Dunn and Taya (1994). We can calculate the nondimensional material constants of PZT-4 as: $f_1 = 5.43, f_2 = 3.04, f_3 = 2.90, f_4 = 4.49, f_5 = 0.84$, $f_6 = -0.34, f_7 = 1.15$ and $f_8 = 1.58$. The stress concentration factor, for example, for the material, of which $f_1 = 1.1 \times 5.43$ and all other constants keep invariant, is 2.405, as may be read off the curve marked f_1 in Fig. 1 where the multiplication factor on the abscissa is 1.1.

In Fig. 2, the variations of $k_\sigma(r) = \sigma_{\theta\theta}(r, 90^\circ, \phi)/\sigma_{zz}^0$ with the radius in the vicinity of the cavity are shown, for four different piezoelectric materials: PZT-4, PZT-5, PZT-7A and $BaTiO_3$, and their corresponding pure elastic ones. Materials of curves from the top to the bottom in the left of Fig. 2 are in turn PZT-4, PZT-5, ..., $BaTiO_3$ and $BaTiO_3$(E), respectively, where (E) indicates the

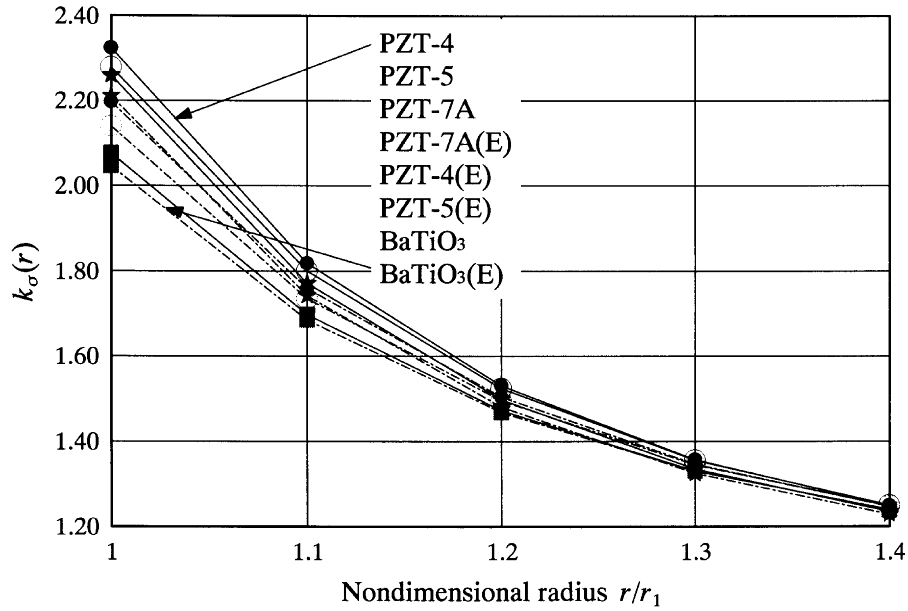

Fig. 2. Variation of $k_\sigma(r)$ in the vicinity of cavity.

pure elastic one corresponding to the specific piezoelectric material. It can be seen that when the radius becomes larger, the difference between the effects of four piezoelectric materials as well as their corresponding elastic ones on the distribution of $k_\sigma(r)$ decreases.

4.5. Electric displacement concentration in the neighborhood of a spherical cavity

At last, let's consider an infinite piezoelastic medium with a spherical cavity ($r = r_1$) due to a uniform electric displacement D_z^0 applied at infinity. The supplementary solution which vanishes at infinity should satisfy the following boundary conditions at $r = r_1$:

$$D_r = -D_z^0 \cos\theta = -P_1(\cos\theta)D_z^0, \quad \sigma_{rr} = \sigma_{r\theta} = 0. \tag{54}$$

By comparing eqns (54) and (44), we get the equations for determining the nonzero constants A_{11}, A_{12} and A_{13} as follows:

$$\sum_{i=1}^{3} A_{1i} r_1^{\nu_{1i}-3/2}\left[c_{33}K_{1i}^1\left(\nu_{1i}-\frac{1}{2}\right)+2c_{13}(K_{1i}^1+1)+\frac{e_{33}^2}{\varepsilon_{33}}K_{1i}^2\left(\nu_{1i}-\frac{1}{2}\right)\right]=0,$$

$$\sum_{i=1}^{3} A_{1i} r_1^{\nu_{1i}-3/2}\left\{c_{44}\left[K_{1i}^1-\left(\nu_{1i}-\frac{3}{2}\right)\right]+\frac{e_{15}e_{33}}{\varepsilon_{33}}K_{1i}^2\right\}=0,$$

$$\sum_{i=1}^{3} A_{1i} r_1^{\nu_{1i}-3/2}\left[e_{33}K_{1i}^1\left(\nu_{1i}-\frac{1}{2}\right)+2e_{31}(K_{1i}^1+1)-e_{33}K_{1i}^2\left(\nu_{1i}-\frac{1}{2}\right)\right]=-D_z^0. \tag{55}$$

The electric displacement concentration factor k_D is defined as

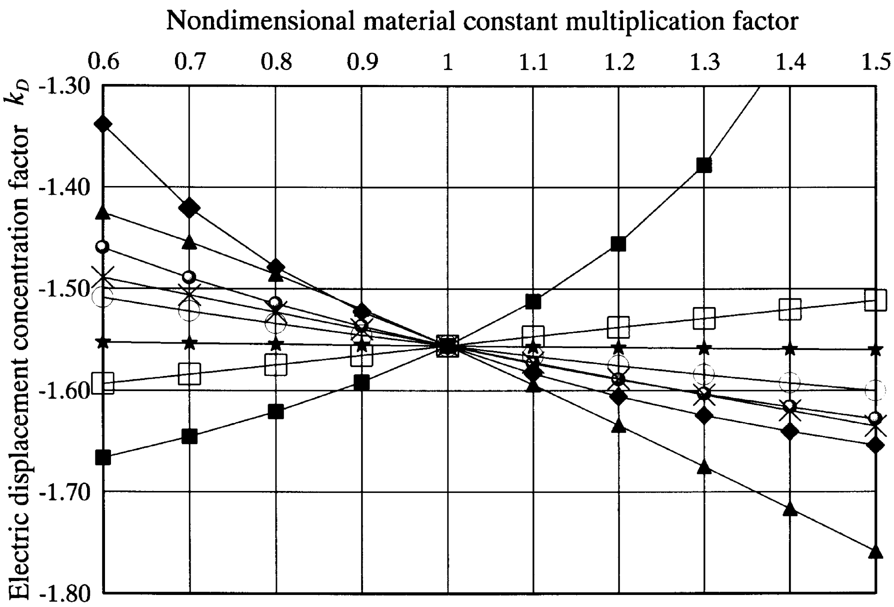

Fig. 3. Electric displacement concentration factor when one of the nondimensional material constants (PZT-4) is multiplied by a factor from 0.6–1.5.

$$k_D = D_\theta(r_1, 90^\circ, \phi)/D_z^0 = -1 - \sum_{i=1}^{3} (e_{33} A_{1i} r_1^{\nu_{1i}-3/2}/D_z^0)\{f_5[K_{1i}^1 - (\nu_{1i} - 3/2)] - f_7 f_8 K_{1i}^2\}. \quad (56)$$

Observe that the electric displacement concentration factor k_D is also independent of the radius r_1 as for the stress concentration factor k_σ.

The effects of material constants on the electric displacement concentration factor k_D are shown in Fig. 3 for PZT-4. It is noted here that curves labeled by the same symbols in Figs. 1 and 3 have identical contents so that they are not marked in Fig. 3. The variations of $k_D(r) = D_\theta(r, 90^\circ, \phi)/D_z^0$ with the radius in the vicinity of the cavity for the four piezoelectric materials are presented in Fig. 4.

5. Conclusions

In this paper, we simplify the basic equations of a radial polarization, spherical isotropic, piezoelastic medium by the introduction of three displacement functions. For the general non-axisymmetric problem, the equations are further reduced to an uncoupled second-order ordinary differential equation in unknown ψ_n, and a coupled system of three second-order ordinary differential equations in the other three unknowns w_n, G_n and Φ_n. It is found that all these second-order differential equations are of the Euler type, to which the solution is well established. Some axisymmetric boundary condition problems are considered based on the general solution: a concentrated force and a point charge acting at the origin of an infinite piezoelectric body, and an infinite

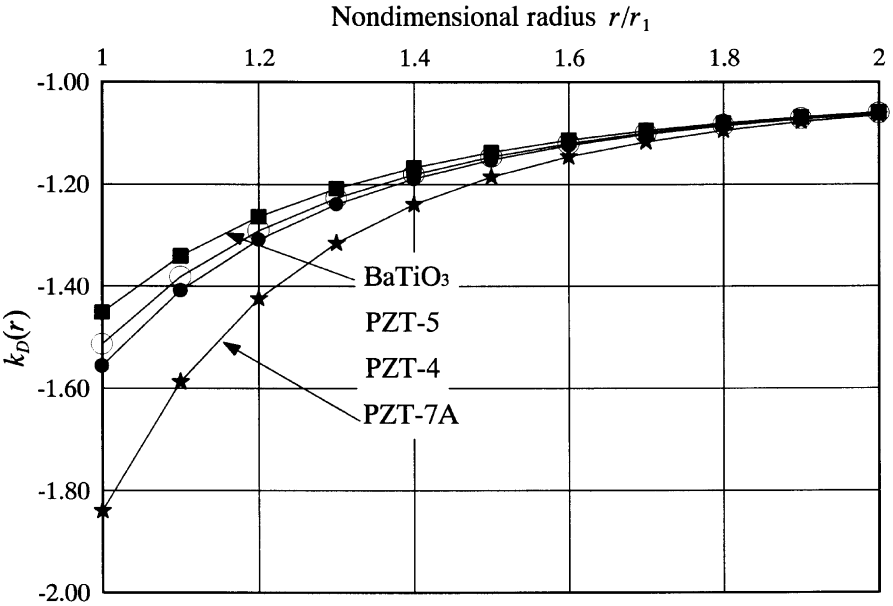

Fig. 4. Variation of $k_D(r)$ in the vicinity of cavity.

piezoelectric medium with a spherical cavity under uniform extension and electric displacement at infinity, respectively. It is shown that some characteristics associated with the pure elasticity still are valid for the piezoelectricity; for example, the stress concentration factor k_σ is independent of the radius r_1 of the cavity. It is also noted that if the piezoelectric effect is neglected, our results degenerate identically to those of the pure elasticity (Chen, 1966). For an infinite piezoelectric body with a spherical cavity, numerical results are presented to demonstrate the effects of material constants on the stress concentration factor k_σ under uniform extension as well as those on the electric displacement concentration factor k_D under uniform electric displacement.

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